

The Inverse Problem of the Calculus of Variations: An Introduction

D. Krupka

The *inverse problem of the calculus of variations* is the problem of finding conditions, ensuring that a given system of (ordinary or partial) differential equations coincides with the system of Euler-Lagrange equations of an integral variational functional. Its origin, dated 1886, is connected with the names of Sonin and Helmholtz; a newer modified version of the inverse problem for systems of ordinary second order equations, using *variational integrating factors*, was presented by Douglas in 1941. Since then the problem, lying on the border of the calculus of variations, mathematical analysis of differential equations, differential geometry, and topology of manifolds was studied by many authors. However, in its generality it still belongs to mathematical problems that wait for a complete solution. The aim of this lecture series is to give an introduction to the local and global inverse problem.

First we consider the variationality problem for systems of ordinary second order differential equations. We derive the Helmholtz variationality conditions and find integrability conditions for the Douglas's problem.

The global inverse problem is then formulated within the global variational theory, extending the classical calculus of variations from Euclidean spaces to smooth manifolds. The problem is to find conditions when a system of equations on a manifold, which is locally variational, admits a *global* Lagrangian. We introduce underlying variational concepts in terms of differential forms, and study the theory of variational sequences, in which one arrow represents the Euler-Lagrange mapping of the calculus of variations. The sequence relates properties of the Euler-Lagrange mapping with the De Rham cohomology of the underlying manifold.

In these lectures we do not consider the inverse problem for vector fields on tangent bundles (sprays), which is related with the Douglas's problem.

Contents

Part 1 The inverse problem for systems of second order ordinary differential equations

1. The inverse problems of Sonin, Helmholtz and Douglas
2. Energy Lagrangians
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4. Variational systems of differential equations and the Helmholtz conditions
5. The Sonin-Douglas's problem
6. The Helmholtz conditions for systems of homogeneous equations

Part 2 The global inverse problem in fibred manifolds

1. Jet structures and differential forms on jet manifolds
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3. Invariant variational principles
4. Variational sequences: The structure of the Euler-Lagrange mapping and the inverse problem
5. The global inverse problem for higher-order fibred mechanics
6. Invariance of the Helmholtz form and the inverse problem

References

The following list includes selected titles, in which further references can be found. It is restricted to sources of mathematical character. Reference [1] contains relatively complete bibliography.

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