18th International Summer School on Global Analysis and its Applications
"The Local and Global Inverse Problem of the Calculus of Variations"
Levoca, Slovakia, 12-17 August 2013
http://www.lepageri.eu/ga2013

A. Bloch and D. Zenkov:

Controlled Lagrangians and the Inverse Problem of the Calculus of Variations

Abstract: The inverse problem of calculus of variations, in its simplest setting, studies when a second order vector field is variational. The answer is given, locally, by the so-called Helmholtz conditions. A controlled vector field is a vector field on a manifold (called state space) that depends on parameters. Control theory studies how to assign these parameters as functions of time and/or state variables in order to accomplish the desired properties of dynamics. For this reason, these parameters are referred to as controls. An important special class of controlled vector fields are mechanical controlled vector fields. These are second order controlled vector fields that are Lagrangian when the controls are set to zero. Of course, specifying the controls generically leads to a non-Lagrangian vector field. It may be, however, interesting and desirable to select controls in such a way that the controlled dynamics is Lagrangian, too. This is the idea behind the method of controlled Lagrangians for stabilization of (relative) equilibria of mechanical systems. For the task to be accomplished, certain matching conditions should be satisfied. Since the controlled dynamics is Lagrangian, it is natural to conjecture that there is a connection between the Helmholtz and matching conditions. We will review the main concepts of the inverse problem of calculus of variations, matching stabilization technique, and elucidate the links between the Helmholtz and matching conditions.